Bernstein-Vandermonde matrices and Applications I

Ana Marco
Departamento de Física y Matemáticas
Universidad de Alcalá. SPAIN

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Outline of the talk

- Bernstein-Vandermonde matrices
- Neville elimination and total positivity
- Bidiagonal factorization
- Linear system solving
- Eigenvalue computation
- Conclusions and references

Vandermonde matrices

Let us consider the **monomial basis** $\{x^i\}_{i=0,\ldots,n}$ of the space $\Pi_n(x)$ of the polynomials of degree less than or equal to n.

$$A = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{l+1} & x_{l+1}^2 & \cdots & x_{l+1}^n \end{pmatrix}$$

is the $(l+1) \times (n+1)$ **Vandermonde matrix** corresponding to the nodes $\{x_i\}_{1 \le i \le l+1}$.

Vandermonde matrices and total positivity

Proposition

If $0 < x_1 < x_2 < \ldots < x_{n+1}$, then the Vandermonde matrix A is **strictly totally positive**.

Definition

A matrix is called **totally positive** (**strictly totally positive**) if all its minors are nonnegative (positive).

Bernstein basis

Bernstein-Vandermonde matrices are a generalization of Vandermonde matrices arising when considering the Bernstein basis (instead of power basis).

Definition

The **Bernstein basis** of the space $\Pi_n(x)$ of the polynomials of degree less than or equal to n on $[0,1]$ is:

$$\mathcal{B}_n = \left\{ b_i^{(n)}(x) = \binom{n}{i}(1-x)^{n-i}x^i, \qquad i = 0, \dots, n \right\}.$$

Example (case n=3)

$$\mathcal{B}_3 = \left\{ \binom{3}{0}(1-x)^3, \binom{3}{1}(1-x)^2 x, \binom{3}{2}(1-x)x^2, \binom{3}{3}x^3 \right\}$$

Bernstein-Vandermonde matrices

Definition

The **Bernstein-Vandermonde matrix** for the Bernstein basis $\mathcal{B}_n$ and the nodes $\{x_i\}_{1\leq i\leq l+1}$ is:

$$A = \begin{pmatrix} \binom{n}{0}(1-x_1)^n & \binom{n}{1}x_1(1-x_1)^{n-1} & \cdots & \binom{n}{n}x_1^n \\ \binom{n}{0}(1-x_2)^n & \binom{n}{1}x_2(1-x_2)^{n-1} & \cdots & \binom{n}{n}x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n}{0}(1-x_{l+1})^n & \binom{n}{1}x_{l+1}(1-x_{l+1})^{n-1} & \cdots & \binom{n}{n}x_{l+1}^n \end{pmatrix}$$

A Bernstein-Vandermonde matrix 5×4

$$A = \begin{pmatrix} \binom{3}{0}(1-x_1)^3 & \binom{3}{1}(1-x_1)^2 x_1 & \binom{3}{2}(1-x_1)x_1^2 & \binom{3}{3}x_1^3 \\ \binom{3}{0}(1-x_2)^3 & \binom{3}{1}(1-x_2)^2 x_2 & \binom{3}{2}(1-x_2)x_2^2 & \binom{3}{3}x_2^3 \\ \binom{3}{0}(1-x_3)^3 & \binom{3}{1}(1-x_3)^2 x_3 & \binom{3}{2}(1-x_3)x_3^2 & \binom{3}{3}x_3^3 \\ \binom{3}{0}(1-x_4)^3 & \binom{3}{1}(1-x_4)^2 x_4 & \binom{3}{2}(1-x_4)x_4^2 & \binom{3}{3}x_4^3 \\ \binom{3}{0}(1-x_5)^3 & \binom{3}{1}(1-x_5)^2 x_5 & \binom{3}{2}(1-x_5)x_5^2 & \binom{3}{3}x_5^3 \end{pmatrix}$$

Vandermonde matrices and total positivity

Proposition

The Bernstein-Vandermonde matrix is **strictly totally positive** when the nodes satisfy $0 < x_1 < x_2 < \ldots < x_{l+1} < 1$ [Carnicer-Peña,93].

Importance I

- The Bernstein basis for $\Pi_n(x)$ is a **widely used basis in CAGD** [Carnicer-Peña, 93; Delgado-Peña, 09; Delgado-Peña, 12; Farin, 02; Farouki, 12].
- The **explicit conversion** between the Bernstein and the power basis is **exponentially ill-conditioned** as the polynomial degree increases [Farouki, 12].
- It is very important that when designing algorithms for performing numerical computations with polynomials in Bernstein form, **all the intermediate operations** are developed **using the Bernstein basis only** [Bini-Gemignani, 04].

Importance II

Consequence

The **accurate and efficient solution of the main problems in Numerical Linear Algebra** (linear system solving, eigenvalue computation, singular value computation and the least squares problem) **for Bernstein-Vandermonde matrices is an essential issue**.

- Linear system solving and eigenvalue computation → In this talk
- The least squares problem and singular value computation → José-Javier Martínez

Ill conditioning

Observation

Bernstein-Vandermonde matrices are **ill conditioned** [Marco-Martínez, 07].

Standard algorithms for solving the problems of linear system solving, eigenvalue computation, singular value computation and the least squares problem, that do not take into account the structure of the Bernstein-Vandermonde matrices give **no accurate results**.

Bidiagonal factorization

THE CLUE

The fast and accurate computation of the **bidiagonal factorization** of a Bernstein-Vandermonde matrix A, allows us to solve in an accurate and efficient way the main problems in Numerical Linear Algebra for A.

THE KEY THEORETICAL TOOL

Neville elimination [Gasca-Peña, 92; 94].

Bidiagonal factorization

Observation

Factorizations in terms of bidiagonal matrices are useful when working with Vandermonde [Björck-Pereyra, 70; Higham, 02], Cauchy [Boros-Kailath-Olshevsky, 99], Cauchy-Vandermonde [Martínez-Peña, 98; 03] or generalized Vandermonde matrices [Demmel-Koev, 05].

Neville elimination

Neville elimination is a type of elimination that makes zeros in a matrix adding to a given row an appropriate multiple of the previous one:

Given $A = (a_{i,j}) \in \mathbf{R}^{n \times n}$, it consists of $n-1$ steps

$$A := A_1 \to A_2 \to \ldots \to A_n$$

where $A_t = (a_{i,j}^{(t)}) \in \mathbf{R}^{n \times n}$ has zeros below its main diagonal in the $t-1$ first columns.

Observation

From now on A is a **square matrix**.

Neville elimination

- $p_{i,j} := a_{i,j}^{(j)}$ **pivot** (i,j) $\quad (1 \leq j \leq n; \quad j \leq i \leq n)$
- If all the pivots are non zero:
  - No row exchanges are needed.
  - $p_{i,1} = a_{i,1} \ \forall i$

$$p_{i,j} = \frac{\det A[i-j+1,\ldots,i|1,\ldots,j]}{\det A[i-j+1,\ldots,i-1|1,\ldots,j-1]} \qquad 1 < j \leq i \leq n$$

[Gasca-Peña, 92].

- $m_{i,j} = \frac{p_{i,j}}{p_{i-1,j}}$ **multiplier** $(1 \leq j \leq n; \quad j < i \leq n)$

Neville elimination

- $U := A_n$ is upper triangular with the **diagonal pivots** $p_{i,i}$ in its main diagonal.
- The **complete Neville elimination** of a A consists of performing the Neville elimination of A for obtaining U and then continue with the Neville elimination of U^T.
- When no row exchanges are needed in the Neville elimination of A and U^T the multipliers of the complete Neville elimination of A are:
  - The multipliers of the Neville elimination of A if $i \geq j$
  - The multipliers of the Neville elimination of A^T if $j \geq i$.

Total positivity

The Neville elimination characterizes the strictly totally positive matrices [Gasca-Peña, 92]:

THEOREM 1

A matrix is **strictly totally positive** if and only if its complete Neville elimination can be performed without row and column exchanges, the multipliers of the Neville elimination of A and A^T are positive, and the diagonal pivots of the Neville elimination of A are positive.

Bernstein-Vandermonde matrices

From now on, A is the **Bernstein-Vandermonde matrix** for the Bernstein basis $\mathcal{B}_n$ and the nodes $\{x_i\}_{1\leq i\leq n+1}$

$$A = \begin{pmatrix} \binom{n}{0}(1-x_1)^n & \binom{n}{1}x_1(1-x_1)^{n-1} & \cdots & \binom{n}{n}x_1^n \\ \binom{n}{0}(1-x_2)^n & \binom{n}{1}x_2(1-x_2)^{n-1} & \cdots & \binom{n}{n}x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n}{0}(1-x_{n+1})^n & \binom{n}{1}x_{n+1}(1-x_{n+1})^{n-1} & \cdots & \binom{n}{n}x_{n+1}^n \end{pmatrix}$$

where the nodes satisfy $0 < x_1 < x_2 < \ldots < x_{n+1} < 1$.

$$\Downarrow$$

A is **strictly totally positive**.

Determinant

Proposition [Marco-Martínez, 07]

Let A be the square Bernstein-Vandermonde matrix of order $n+1$ for the Bernstein basis $\mathcal{B}_n$ and the nodes $x_1, x_2, \ldots, x_{n+1}$.

$$\det A = \binom{n}{0}\binom{n}{1}\cdots\binom{n}{n}\prod_{1\leq i<j\leq n+1}(x_j - x_i).$$

Determinant

Corollary 1

$$\det \begin{pmatrix} (1-x_1)^n & x_1(1-x_1)^{n-1} & \cdots & x_1^n \\ (1-x_2)^n & x_2(1-x_2)^{n-1} & \cdots & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ (1-x_{n+1})^n & x_{n+1}(1-x_{n+1})^{n-1} & \cdots & x_{n+1}^n \end{pmatrix} = \prod_{1\le i<j\le n+1} (x_j - x_i)$$

Bidiagonal factorization of A^{-1}

THEOREM 2 [Marco-Martínez, 07]

Let $A \in \mathbf{R}^{(n+1)\times(n+1)}$ be a Bernstein-Vandermonde matrix for $\mathcal{B}_n$ whose nodes satisfy $0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < 1$. Then

$$A^{-1} = G_1 G_2 \cdots G_n D^{-1} F_n F_{n-1} \cdots F_1$$

where G_i are upper triangular bidiagonal matrices, F_i are lower triangular bidiagonal matrices ($i = 1, \ldots, n$), and D is a diagonal matrix.

Bidiagonal factorization of A^{-1}

$$F_i = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & -m_{i+1,i} & 1 & & & \\ & & & & -m_{i+2,i} & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & -m_{n+1,i} & 1 \end{pmatrix},$$

$m_{i,j}$ are the multipliers of the Neville elimination of A.

Bidiagonal factorization of A^{-1}

$$G_i^T = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & -\widetilde{m}_{i+1,i} & 1 & & & \\ & & & & -\widetilde{m}_{i+2,i} & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & -\widetilde{m}_{n+1,i} & 1 \end{pmatrix},$$

$\widetilde{m}_{i,j}$ are the multipliers of the Neville elimination of A^T.

Bidiagonal factorization of A^{-1}

$$D = \begin{pmatrix} p_{11} & & & \\ & p_{22} & & \\ & & \ddots & \\ & & & p_{n+1,n+1} \end{pmatrix}$$

$p_{i,i}$ are the diagonal pivots of the Neville elimination of A.

Bidiagonal factorization of A^{-1}

- $m_{i,j} = \dfrac{(1-x_i)^{n-j+1}(1-x_{i-j})\prod_{k=1}^{j-1}(x_i-x_{i-k})}{(1-x_{i-1})^{n-j+2}\prod_{k=2}^{j}(x_{i-1}-x_{i-k})}$,

  where $j = 1, \ldots, n; i = j+1, \ldots, n+1$.

- $\widetilde{m}_{i,j} = \dfrac{(n-i+2)x_j}{(i-1)(1-x_j)}, \qquad j = 1, \ldots, n; \; i = j+1, \ldots, n+1.$

- $p_{i,i} = \dfrac{\binom{n}{i-1}(1-x_i)^{n-i+1}\prod_{k<i}(x_i-x_k)}{\prod_{k=1}^{i-1}(1-x_k)}, \qquad i = 1, \ldots, n+1$

Bidiagonal factorization of A^{-1}

SKETCH OF THE PROOF of THEOREM 2

A STP

$\Downarrow$ THEOREM 1

complete Neville elimination without row and column exchanges

$\Downarrow$ [Gasca-Peña,92; 94]

$$A^{-1} = G_1 G_2 \cdots G_n D^{-1} F_n F_{n-1} \cdots F_1$$

where F_i, G_i and D are the matrices in the statement of this theorem

Bidiagonal factorization of A^{-1}

SKETCH OF THE PROOF of THEOREM 2

Neville elimination without row exchanges

$$\Downarrow$$

$$p_{i,j} = \frac{\det A[i-j+1,\ldots,i|1,\ldots,j]}{\det A[i-j+1,\ldots,i-1|1,\ldots,j-1]}, \qquad 1 < j \leq i \leq n.$$

OBS: $m_{i,j} = \frac{p_{i,j}}{p_{i-1,j}}, \quad 1 \leq j \leq n; \quad j < i \leq n.$

Bidiagonal factorization of A^{-1}

SKETCH OF THE PROOF of THEOREM 2

Properties of the determinants + Corollary 1

$$\Downarrow$$

$$\det A[i,\dots,i+j-1|1,\dots,j] = \binom{n}{0}\binom{n}{1}\cdots\binom{n}{j-1}$$
$$(1-x_i)^{n-j+1}(1-x_{i+1})^{n-j+1}\cdots(1-x_{i+j-1})^{n-j+1}\prod_{i\leq k<l\leq i+j-1}(x_l-x_k),$$

Bidiagonal factorization of A^{-1}

SKETCH OF THE PROOF of THEOREM 2

Simplifying:

$$m_{i,j} = \frac{(1-x_i)^{n-j+1}(1-x_{i-j})\prod_{k=1}^{j-1}(x_i - x_{i-k})}{(1-x_{i-1})^{n-j+2}\prod_{k=2}^{j}(x_{i-1} - x_{i-k})},$$

where $j = 1, \ldots, n;\ i = j+1, \ldots, n+1$.

OBS: Analogously we prove the expressions for $p_{i,i}$ and $\widetilde{m}_{i,j}$ in the statement of Theorem 2.

Bidiagonal factorization of A

THEOREM 3 [Marco-Martínez, 13]

Let $A \in \mathbf{R}^{(n+1)\times(n+1)}$ be a Bernstein-Vandermonde matrix for $\mathcal{B}_n$ whose nodes satisfy $0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < 1$. Then

$$A = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n \tag{3.1}$$

where G_i are order $n+1$ upper triangular bidiagonal matrices $(i = 1, \ldots, n)$, F_i are order $n+1$ lower triangular bidiagonal matrices $(i = 1, \ldots, n)$, and D is an order $(n+1)$ diagonal matrix.

Bidiagonal factorization of A

$$F_i = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & m_{i+1,1} & 1 & & & \\ & & & & m_{i+2,2} & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & m_{n+1,n+1-i} & 1 \end{pmatrix},$$

$m_{i,j}$ are the multipliers of the Neville elimination of A.

Bidiagonal factorization of A

$$G_i^T = \begin{pmatrix} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & 0 & 1 & & & & & \\ & & & \widetilde{m}_{i+1,1} & 1 & & & & \\ & & & & \widetilde{m}_{i+2,2} & 1 & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & \widetilde{m}_{n+1,n+1-i} & 1 \end{pmatrix},$$

$\widetilde{m}_{i,j}$ are the multipliers of the Neville elimination of A^T.

Bidiagonal factorization of A

$$D = diag\{p_{1,1}, p_{2,2}, \ldots, p_{n+1,n+1}\} \in \mathbf{R}^{(n+1)\times(n+1)}.$$

$p_{i,i}$ are the diagonal pivots of the Neville elimination of A.

Remarks

Remark 1

The formulae we obtain for $m_{i,j}$, $\widetilde{m}_{i,j}$ and $p_{i,i}$ in THEOREM 2 and THEOREM 3 are the same. That is, the Neville elimination of A gives the bidiagonal decomposition of A and A^{-1}.

Remark 2

In the square case, the bidiagonal matrices F_i and G_i $(i = 1, \ldots, n)$ that appear in the bidiagonal factorization of A^{-1} are not the same as the ones that appear in the bidiagonal factorization of A, nor their inverses. Obtaining the bidiagonal factorization of A from the bidiagonal factorization of A^{-1} (or vice versa) is not straightforward [Gasca-Peña, 96].

Algorithm TNBDBV: pseudocode

Computation of the $m_{i,j}$:

for $i = 2 : n+1$
  $M = \frac{(1-x_i)^n}{(1-x_{i-1})^{n+1}}$
  $m_{i,1} = (1 - x_{i-1}) \cdot M$
  for $j = 1 : i-2$
    $M = \frac{(1-x_{i-1})(x_i - x_{i-j})}{(1-x_i)(x_{i-1} - x_{i-j-1})} \cdot M$
    $m_{i,j+1} = (1 - x_{i-j-1}) \cdot M$
  end
end

Computation of the $\widetilde{m}_{i,j}$:

for $j = 1 : n$
  $c_j = \frac{x_j}{1-x_j}$
  for $i = j+1 : n+1$
    $\widetilde{m}_{i,j} = \frac{n-i+2}{i-1} \cdot c_j$
  end
end

Algorithm TNBDBV: pseudocode

Computation of the $p_{i,i}$ of D:

$$
\begin{array}{l}
q = 1 \\
p_{1,1} = (1 - x_1)^n \\
\texttt{for } i = 1 : n \\
\quad q = \frac{(n-i+1)}{i(1-x_i)} \cdot q \\
\quad aux = 1 \\
\quad \texttt{for } k = 1 : i \\
\quad\quad aux = (x_{i+1} - x_k) \cdot aux \\
\quad \texttt{end} \\
\quad p_{i+1,i+1} = q \cdot (1 - x_{i+1})^{n-i} \cdot aux \\
\texttt{end}
\end{array}
$$

Algorithm TNBDBV: comments

- Our algorithm TNBDBV for computing the bidiagonal decomposition of A by using the formulae we have presented in this section for $m_{i,j}$, $\widetilde{m}_{i,j}$ and $p_{i,i}$ has been presented in [Marco-Martínez, 07].
- The Bernstein-Vandermonde matrix A is not constructed.
- Its implementation in MATLAB can be obtained from the package **TNTool** of P. Koev (http://www-math.sjsu.edu/∼koev).
- The output of TNBDBV is a $(n+1) \times (n+1)$ matrix, $\mathcal{BD}(A)$, containing $m_{i,j}$, $\widetilde{m}_{i,j}$ and $p_{i,i}$.
- Computational cost: $O(n^2)$ ops.
- It has high relative accuracy (avoids substractive cancellation).

Error analysis of TNBDBV

THEOREM 4 [Marco-Martínez, 13] square case

Let:

- A be a strictly totally positive Bernstein–Vandermonde matrix for the Bernstein basis $\mathcal{B}_n$ and the nodes $\{x_i\}_{1\leq i\leq n+1}$.
- $\mathcal{BD}(A) = (b_{i,j})_{1\leq i\leq n+1}$ be the matrix representing the exact bidiagonal decomposition of A.
- $(\widehat{b}_{i,j})_{1\leq i\leq n+1}$ be the matrix representing the computed bidiagonal decomposition of A by means of the algorithm TNBDBV in floating point arithmetic with machine precision ϵ.

Then

$$|\widehat{b}_{i,j} - b_{i,j}| \leq \frac{(4n^2+2n)\epsilon}{1-(4n^2+2n)\epsilon} b_{i,j}, \quad i,j = 1,\ldots,n+1.$$

Error analysis of TNBDBV: comments

- We use the **standard model of floating point arithmetic** [Higham, 02]:

  Let x, y be floating point numbers and ϵ be the machine precision,

  $$fl(x \odot y) = (x \odot y)(1+\delta)^{\pm 1}, \quad \text{where } |\delta| \leq \epsilon, \ \odot \in \{+, -, \times, /\}.$$

- The errors are accumulated in the style of Higham (Chapter 3 of [Higham, 02]).
- The expression in THEOREM 4 is the error bound for computing the $m_{i,j}$. The error bounds for computing the $\widetilde{m}_{i,j}$ and the $p_{i,i}$ are lower.

Bidiagonal factorization: perturbation theory

AIM: Let A be a BV matrix. To prove:

Small relative perturbations in the nodes of A

$\Downarrow$

Small relative perturbations in its bidiagonal factorization $\mathcal{BD}(A)$

Bidiagonal factorization: perturbation theory

Definitions

Let A be a strictly totally positive BV matrix for the Bernstein basis $\mathcal{B}_n$ and the nodes x_i, and let $x_i' = x_i(1+\delta_i)$ be the perturbed nodes for $1 \leq i \leq n+1$, where $|\delta_i| << 1$.

- $rel\_gap_x \equiv \min_{i \neq j} \frac{|x_i - x_j|}{|x_i| + |x_j|}$
- $rel\_gap_1 \equiv \min_i \frac{|1 - x_i|}{|x_i|}$
- $\theta \equiv \max_i \frac{|x_i - x_i'|}{|x_i|} = \max_i |\delta_i|$ (greatest relative perturbation)
- $\alpha \equiv min\{rel\_gap_x, rel\_gap_1\}$
- $\kappa_{BV} \equiv \frac{1}{\alpha}$

where $\theta << rel\_gap_x, rel\_gap_1$.

Bidiagonal factorization: perturbation theory

THEOREM 5: [Marco-Martínez, 13]

Let:

- A be a strictly totally positive BV matrix for the Bernstein basis $\mathcal{B}_n$ and the nodes x_i.
- A' be a strictly totally positive BV matrix for the Bernstein basis $\mathcal{B}_n$ and the perturbed nodes $x_i' = x_i(1+\delta_i)$.
- $\mathcal{BD}(A)$ the bidiagonal decomposition of A.
- $\mathcal{BD}(A')$ the bidiagonal decomposition of A'.

Then:

$$|(\mathcal{BD}(A'))_{i,j} - (\mathcal{BD}(A))_{i,j}| \leq \frac{(2n+2)\kappa_{BV}\theta}{1-(2n+2)\kappa_{BV}\theta}(\mathcal{BD}(A))_{i,j}.$$

Bidiagonal factorization: perturbation theory

Comments

- The perturbations are accumulated in the style of Higham [Higham, 02]
- $(2n+2)\kappa_{BV}$ is an appropriate structured condition number of A with respect to the relative perturbations in the data x_i.

  Relevant quantities for the determination of an structured condition number are the **relative separations between the nodes** (in our case also the **relative distances to 1**). Analogous results in [Koev, 05; Demmel-Koev, 06].

Algorithm. $O(n^2)$.

AIM: To **solve** $Ax = b$ where $A \in \mathbf{R}^{(n+1)\times(n+1)}$ is a Bernstein-Vandermonde matrix.

INPUT: The nodes x_i $(i = 1, \ldots, n+1)$ and the vector b.
OUTPUT: The solution vector x.

Step 1: Computation of the bidiagonal decomposition of A^{-1} by using `TNBDBV`.
Step 2: Computation of

$$x = A^{-1}b = G_1 G_2 \cdots G_n D^{-1} F_n F_{n-1} \cdots F_1 b$$

by using `TNSolve` (package TNTool of P. Koev).

Numerical experiments

We consider:

- The Bernstein basis $\mathcal{B}_{15}$.
- The Bernstein-Vandermonde matrix $A \in \mathbf{R}^{16\times 16}$ generated by $\frac{1}{18} < \frac{1}{16} < \frac{1}{14} < \frac{1}{12} < \frac{1}{10} < \frac{1}{8} < \frac{1}{6} < \frac{1}{4} < \frac{11}{20} < \frac{19}{34} < \frac{17}{30} < \frac{15}{26} < \frac{11}{18} < \frac{9}{14} < \frac{7}{10} < \frac{5}{6}$.
- $b_1 = (2, 1, 2, 3, -1, 0, 1, -2, 4, 1, 1, -3, 0, -1, -1, 2)^T$
- $b_2 = (1, -2, 1, -1, 3, -1, 2, -1, 4, -1, 2, -1, 1, -3, 1, -4)^T$

We compute the **relative error** of a solution x by means of:

$$err = \frac{\|x - x_e\|_2}{\|x_e\|_2}.$$

x_e is the exact solution computed in *Maple*.

Numerical experiments

We solve the two linear systems by means of:

- TNBDBV: our algorithm.
- TNBD: classical Neville elimination.
- $A \backslash b$ MATLAB command: Gaussian elimination.

b_i	*TNBDBV*	*TNBD*	$A \backslash b$
b_1	1.0e-15	5.9e-11	6.5e-12
b_2	4.9e-16	5.9e-11	6.4e-12

Table: Relative errors

The condition number of A is: $\kappa_2(A) = 3.4e + 09$.

Applications: Implicitization of curves

The solution of linear systems whose coefficients matrices are Bernstein-Vandermonde matrices is required in the solution of the problem:

Given a **plane curve** by means of its parametric equations **in Bernstein form** (the usual situation in the case of Bézier curves), computing by using **resultants** and **interpolation**, and avoiding basis conversion between Bernstein and monomial basis, its **implicit equation** in the bivariate tensor-product Bernstein basis.

More information: A. Marco, J. J. Martínez, Bernstein-Bezoutian matrices and curve implicitization, Theoretical Computer Science 377(2007) 65–72.

Algorithm. $O(n^3)$.

AIM: To **compute the eigenvalues** of a Bernstein-Vandermonde matrix $A \in \mathbf{R}^{(n+1)\times(n+1)}$.

INPUT: The nodes x_i $(i = 1, \ldots, n+1)$.
OUTPUT: A vector x containing the eigenvalues of A.

Step 1: Computation of the bidiagonal decomposition of A by using `TNBDBV`.
Step 2: Given the result of Step 1, computation of the eigenvalues of A by using `TNEigenvalues` ([Koev, 05]; package TNTool).

Numerical experiments

We consider:

- The Bernstein basis $\mathcal{B}_{20}$.
- The Bernstein-Vandermonde matrix $A \in \mathbf{R}^{21\times 21}$ generated by $\frac{1}{12} < \frac{1}{11} < \frac{1}{10} < \frac{1}{9} < \frac{1}{8} < \frac{1}{7} < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{1}{2} < \frac{7}{12} < \frac{13}{22} < \frac{3}{5} < \frac{11}{18} < \frac{5}{8} < \frac{9}{14} < \frac{2}{3} < \frac{7}{10} < \frac{3}{4} < \frac{5}{6}$.

The condition number of A is: $\kappa_2(A) = 1.9e+12$.

We compute the **relative error** of each computed eigenvalue by using the exact eigenvalues calculated in *Maple*.

Numerical experiments

We present **the two greatest relative errors** obtained when computing the eigenvalues of A by means of:

- Our algorithm:
  - $2.8e-15$ (18th eigenvalue).
  - $2.1e-15$ (20th eigenvalue).
- eig from MATLAB:
  - $1.0e-05$ (21st eigenvalue).
  - $6.4e-08$ (20th eigenvalue).

OBS: We consider the eigenvalues sorted from the largest to the smallest one.

Conclusions

Although the problems of **linear system solving** and **eigenvalue computation** for a totally positive Bernstein-Vandermonde matrix can be solved by using standard methods, the algorithms that exploit the structure of the matrix give much more accurate results when the condition number of this matrix is high.

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